Information processing of a complex system: Response behaviors specified by the Weierstrass function

Hiroaki Hara, Tunehiro Obata, and Junji Koyama
Department of Information Science, Graduate School of Tohoku University, Sendai 980, Japan and Geophysical Institute, Tohoku University, Aramaki-Aoba, Sendai 980, Japan
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By generalizing the usual lateral neural inhibition, analytic behavior of the response function of a complex system is studied. Units of the system are assumed to be interconnected vertically by successive bifurcations and influenced horizontally by lateral inhibitory couplings. It is found that the response function is approximately expressed by the Weierstrass (W) function and the asymptotic behavior is specified by a power law. The exponent of the power law represents the fractal dimension determined by a geometrical structure of the interconnections and the lateral inhibitory couplings.

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I. INTRODUCTION

Information processing of complex systems has been of great importance for studying fundamental properties of the biological neural network system [1] and other systems [2,3]. Originally, the problem of information processing had been studied by looking for a relationship between information and entropy [4]. Among other things, the visual pattern processing by the biological systems is well known; the Mach bands and all similar phenomena have been explained by the lateral neural interactions between the constituent units. Mathematical models are based on experiments on the retina of limulus by Ratliff, Hartline, and Miller [5], and on the psychophysical data about human vision [6]. The models contain a few simply interconnected layers (receptors, intermediate units, and output units). The response properties of the models have been studied in terms of propagator functions expressed by a Fourier series [7,8].

Generally, the main functions of neural network systems are understood by the geometrical structure due to the connections among the units. No power laws have been mentioned in the visual pattern processing as the response properties. The power laws are fundamental concepts which are related to the scaling invariant sets providing the self-similarity [9,10]. The Weierstrass (W) function representing the self-similarity shows nonstandard behavior. Recently, properties of the W function have been studied by the dynamics in a complex system [11]. Related nonstandard response behaviors have been simulated by analog electrical circuits [12]. It is an interesting problem to consider the W function by a model circuit from an architectural point of view.

In the present paper, we propose a model of information processing of a complex system. Specifically, we study behavior of the response function representing a visual pattern processing. Constituent units are interconnected vertically by successive bifurcations and influenced horizontally by lateral couplings. The response function is approximately expressed by the Weierstrass (W) function, in contrast to the usual response function expressed by a Fourier series [7,8]. The asymptotic form of the W function results in a power law specified by the fractal dimension of the system. Parameters of the W function specify vertical connections and horizontal inhibitory couplings.

II. COMPLEX NEURAL NETWORK SYSTEM

We consider a visual pattern processing. The system is a complex neural network system (see Fig. 1) composed of input (receptor) units, intermediate units, and output units. The constituent units are interconnected vertically by successive bifurcations with the receptor units designated by indices $k=0,\pm 1,\pm 2,\ldots,\pm(M-1)$. The receptor and intermediate units are influenced horizontally by lateral inhibitory couplings.

Suppose that the units located at site k and level 0 receive external stimuli X_k. Local stimuli from the top units to the intermediate or the bottom ones with index k are diminished by the inhibitory influences exerted by other units. To depict the connections, we show a neural network specified by two bifurcated fibers in Figs. 1(a) and 1(b), respectively.

We start with a case where each unit on a site identified by index k influences those on neighboring sites, identified by indices $k-1$ and $k+1$, respectively. The indices k stand for sites distributed over points $-(M-1),-(M-2),\ldots,-1,0,1,\ldots,(M-2),(M-1)$. Later, we will take a limit $M\rightarrow\infty$. The influences are expressed by the lateral inhibitory couplings $g_1^{(l)}$, indicating the interactions between local input units $y_k^{(l)}$ and $y_{k\pm 1}^{(l)}$ located at the lth level ($l=0,1,2,\ldots,L$). The influences by other units will be considered in Sec. III.

The lateral inhibitory couplings $g_1^{(l)}$ are assumed to be very small; they decrease with increasing level index l. The local responses of the units with index k, $y_k^{(l)}$, at level l are functions of the local stimuli $x_k^{(l)}$ and the couplings between the local responses $y_{k\pm 1}^{(l)}$ [see Fig. 1(b)],

$$y_k^{(l)}=x_k^{(l)}-g_1^{(l)}(y_{k+1}^{(l)}+k_{k-1}^{(l)}) \quad (l=0,1,2,\ldots,L)\,, \tag{2.1}$$

for $k=0,\pm1,\pm2,\ldots,\pm(M-1)$, where $1>g_1^{(0)}>g_1^{(1)}>\cdots>g_1^{(L)}>0$ and external stimuli $x_k^{(0)}=X_k$.

Here we suppose that at each vertex $\varphi_k^{(l)}$ there hold relations between the local stimuli and the local responses,

$$bx_k^{(l)}=\varphi_k^{(l)}y_k^{(l-1)} \quad (l\geq 1)\ , \tag{2.2}$$

including the phases, where b is a factor specifying the number of bifurcated fibers [see $b=2$ or $b=1$ in Fig. 1(c)]. The factor will be extended later to a real number b. The real number b (<1) represents a weight factor of conduction in a single fiber. The relations (2.2) that we require are expressed differently by two statements:

(i) The vertex parts $\varphi_k^{(l)}$ regulate an amplitude of the local response $y_k^{(l-1)}$ to be equal to the amplitude of the local stimuli $x_k^{(l)}$ multiplied by b.

(ii) During the successive local stimuli-response propagations from level 0 to level l or $l-1$, the corresponding phase Θ_k on site k is proportional to a number (factor) of bifurcations b^l or b^{l-1}; $\Theta_k=\Phi_k b^l$ or $\Phi_k b^{l-1}$, where Φ_k are proportional coefficients parametrized by sites k.

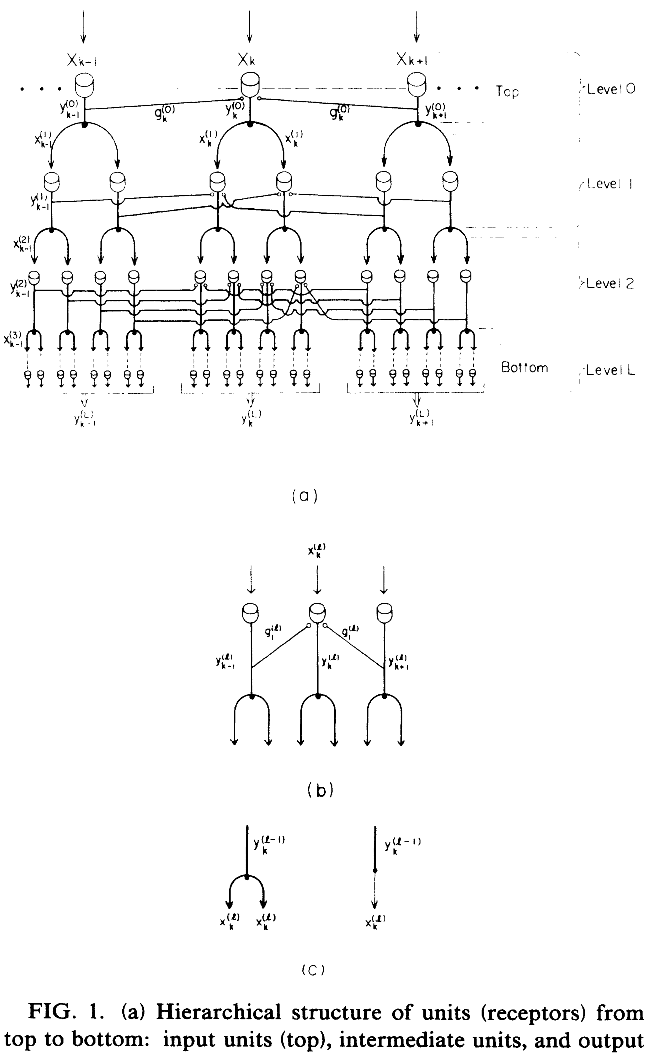

FIG. 1. (a) Hierarchical structure of units (receptors) from top to bottom: input units (top), intermediate units, and output units (bottom). The units are interconnected vertically by successive bifurcated fibers, and the units are also influenced horizontally by lateral inhibitory couplings between the units. (b) Local stimuli $x_k^{(l)}$ influenced horizontally by the inhibition couplings. (c) Vertex part, expressed by a black circle, for $b=2$ or $b=1$, representing an element regulating the phase differences given by Eqs. (2.2) and (2.3).

Then, based on the two statements, we are able to associate the vertex part $\varphi_k^{(l)}$ with the phase differences between levels l and $l-1$ on sites k [$=0,\pm1,\pm2,\ldots,\pm(M-1)$] as follows:

$$\varphi_k^{(l)}=\frac{e^{ikb^{l-1}\theta}}{e^{ikb^l\theta}} \tag{2.3}$$

when we consider a specialized case in which the site $k=0$ has no phase difference and the other sites have increasing phase differences designated by $\Phi_k=\theta|k|$.

The relations (2.2) with (2.3) can be simulated by setting up suitable analog electrical circuits. Note that the variable θ in (2.3) is a complex number and the imaginary part of θ denotes an energy dissipated at the vertex. The energy dissipated is the energy loss arising from the phase differences between the levels.

To simplify the notations of (2.3), we introduce symbols z_l associated with the phase factors,

$$z_l=e^{ib^l\theta}\ , \tag{2.4}$$

and then we rewrite (2.3) as

$$\varphi_k^{(l)}=\frac{z_{l-1}^k}{z_l^k}\ . \tag{2.5}$$

Multiplying (2.1) by z_l^k and summing over k, we obtain

$$\begin{aligned}Y^{(l)}(z_l)&=X^{(l)}(z_l)-2g_1^{(l)}\left[\frac{z_l^{-1}+z_l}{2}\right]Y^{(l)}(z_l)\\&=X^{(l)}(z_l)-2g_1^{(l)}\cos\theta_l\,Y^{(l)}(z_l)\\&\qquad\qquad (l=0,1,2,\ldots,L)\ ,\end{aligned} \tag{2.6}$$

where

$$\theta_l\equiv b^l\theta \tag{2.7}$$

and the symbols $Y^{(l)}(z_l)$ and $X^{(l)}(z_l)$ are generating functions defined by

$$\begin{aligned}Y^{(l)}(z_l)&=\sum_{k=-\infty}^{\infty} z_l^k y_k^{(l)}\ ,\\X^{(l)}(z_l)&=\sum_{k=-\infty}^{\infty} z_l^k x_k^{(l)}\ ,\end{aligned} \tag{2.8}$$

where we have taken the limit $M\to\infty$ in both ends of site $k=\pm(M-1)$. Similarly, we obtain expressions

$$bX^{(l)}(z_l)=Y^{(l-1)}(z_{l-1}) \quad (l=1,2,\ldots,L) \tag{2.9}$$

from the expressions (2.2) with (2.5).

With the aid of (2.6) and (2.9), we obtain basic relationships between the functions $Y^{(l)}(z_l)$,

$$Y^{(L)}(z_L)=\frac{1}{1+2g_1^{(L)}\cos\theta_L}\frac{1}{b}Y^{(L-1)}(z_{L-1})$$
$$=\prod_{l=1}^{L}\left[\left[\frac{1}{b}\right]\frac{1}{1+2g_1^{(l)}\cos\theta_l}\right]Y^{(0)}(z_0)$$
$$=b\prod_{l=0}^{L}\left[\left[\frac{1}{b}\right]\frac{1}{1+2g_1^{(l)}\cos\theta_l}\right]X^{(0)}(z_0)$$
$$(z_l=e^{i\theta_l}\ ,\quad \theta_l=b^l\theta)\ , \qquad (2.10)$$

that is,

$$b^L Y^{(L)}(z_L)\equiv\frac{1}{F_L(\theta;g_1,b)}X^{(0)}(z_0)\ , \qquad (2.11)$$

where

$$F_L(\theta;g_1,b)=\prod_{l=0}^{L}[1+2g_1^{(l)}\cos\theta_l]\ . \qquad (2.12)$$

Since the inhibitory couplings $g_1^{(l)}$ are very small values, we may approximate the expression in the denominator of $b^L Y^{(L)}(z_L)$ by

$$F_L(\theta;g_1,b)=1+2\left[\sum_{l=0}^{L}g_1^{(l)}\cos\theta_l\right]+O(g_1^{(i)}g_1^{(j)})\ , \qquad (2.13)$$

where indices i and j stand for the levels l. The smallness of $g_1^{(i)}$ allows us to neglect the contributions following the second term of (2.13). Note that the function $F_L(\theta;g_1,b)$ denotes the response of the units, constructed as shown in Fig. 1(a). In what follows, we call $F_L(\theta;g_1,b)$ a response function.

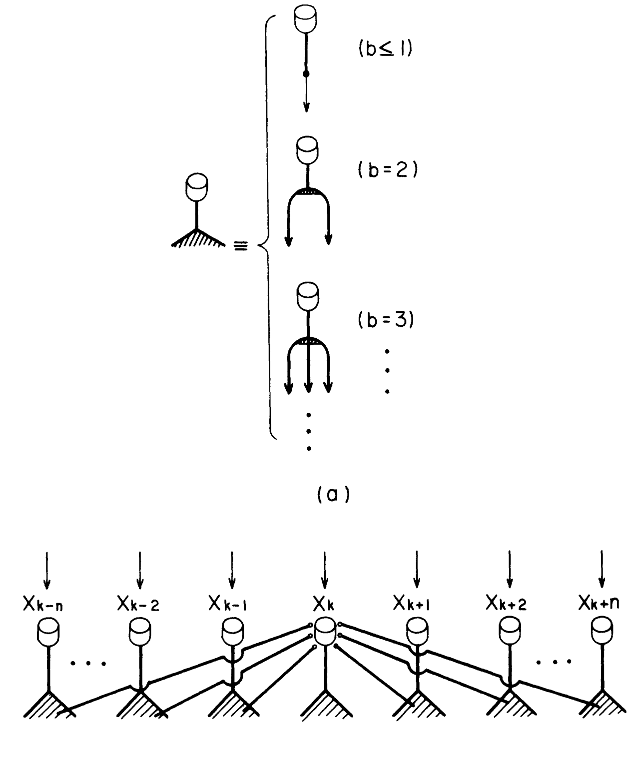

FIG. 2. (a) Skeleton diagrams: the shaded vertex parts at the right-hand side represent the renormalized vertex part for a set of bare vertices located at the given site. (b) Extension of range of the inhibitory couplings [see (2.1) and (3.1)].

III. LOCAL RESPONSE FUNCTION

What has been mentioned in Secs. I and II can be modified. The other lateral influences will be considered, besides the nearest-neighboring lateral inhibitory couplings $g_1^{(l)}$. In this procedure, we extend a range of the inhibitory lateral influences so that the units having indices $k\pm n$ with $n=1,2,\ldots,N(>1)$ are involved. We now have to consider not only the lateral inhibitory couplings $g_1^{(l)}$ but the couplings $g_n^{(l)}$ up to $g_N^{(l)}$ (see Fig. 2). The couplings $g_n^{(l)}$ link $y_k^{(l)}$ to $y_{k+n}^{(l)}$ and to $y_{k-n}^{(l)}$. An expression, corresponding to (2.1), is given by

$$y_k^{(l)}=x_k^{(l)}-\sum_{n=1}^{N}g_n^{(l)}(y_{k+n}^{(l)}+y_{k-n}^{(l)}) \qquad (3.1)$$

for $l=0,1,2,\ldots,L$, and $x_k^{(0)}=X_k$.

The same procedures that we adopted in obtaining (2.11) now lead us to a more general expression,

$$b^L Y^{(L)}(z_L)\equiv\frac{1}{F_{N,L}(\theta;\{g_n\},b)}X^{(0)}(z_0)\ , \qquad (3.2)$$

with

$$F_{N,L}(\theta;\{g_n\},b)=\prod_{l=0}^{L}\left[1+2\sum_{n=1}^{N}g_n^{(l)}\cos n\theta_l\right]\ . \qquad (3.3)$$

If we approximate (3.3) as follows,

$$F_{N,L}(\theta;\{g_n\},b)\approx 1+2\sum_{n=1}^{N}\sum_{l=0}^{L}g_n^{(l)}\cos nb^l\theta$$
$$=\sum_{n=0}^{N}f_n(\theta,L)$$
$$\left[f_0(\theta,L)=1\ ,\quad f_n(\theta,L)=2\sum_{l=0}^{L}g_n^{(l)}\cos nb^l\theta\right]\ , \qquad (3.4)$$

then we note that the last expression $f_n(\theta,L)$ is identical to the Weierstrass (W) function. The function $f_n(\theta,L)$ will be called a local response function. It is known that the W function is continuous everywhere, but is nowhere differentiable [9,10]. Taking into account that the inhibitory couplings $g_n^{(l)}$ become smaller with an increase of the level index l, we introduce a parameter a_n defined by

$$g_n^{(1)}=a_n<1\ . \qquad (3.5)$$

By choosing suitable values of the inhibitory couplings, we suppose that $g_n^{(l)}$ can be rewritten in a form,

$$g_n^{(l)}=[g_n^{(1)}]^l=a_n^l\quad (N\geq n\geq 1)\ . \qquad (3.6)$$

By replacing the variable θ by $b\theta$ in the local response function of (3.4), we obtain

$$f_n(b\theta,\infty)=2\sum_{l=0}^{\infty}a_n^l\cos nb^{l+1}\theta$$
$$=\frac{1}{a_n}[f_n(\theta,\infty)-2\cos n\theta]\ . \qquad (3.7)$$

The last expression shows that there holds an approximate relation,

$$f_n(\theta,\infty)\sim a_n f_n(b\theta,\infty)\ , \qquad (3.8)$$

where we have neglected the contribution $2\cos n\theta$ arising from $l=0$ in the first expression of (3.7). Therefore, one finally gets the following asymptotic form of $f_n(\theta,\infty)$ [14,15] from (3.8):

$$f_n(\theta,\infty)\sim\theta^{-\gamma_n}\ , \qquad (3.9)$$

where γ_n is an n dependent exponent of the power law,

$$\gamma_n=\frac{\ln a_n}{\ln b}\ . \qquad (3.10)$$

Here it is important to note that the fractal dimension γ_n is determined by the lateral inhibitory couplings and the successive bifurcations.

IV. BEHAVIOR OF RESPONSE FUNCTION

In this section we derive approximately an analytic form of the response function $F_{N,L}(\theta;\{g_n\},b)$ [see (3.2)] for $b<1$, as $L\to\infty$ and $N\to\infty$. In this case, b (<1) may be regarded as a weight factor to the local stimuli as mentioned before (see Fig. 3), whereas the case that b is an integer represents the number of bifurcations.

Here we consider a response function defined by

$$\psi(\theta;\{a_n\},b)\equiv\lim_{N,L\to\infty}F_{N,L}(\theta;\{g_n\},b)$$
$$(g_n^{(l)}=[g_n^{(1)}]^l=a_n^l)\ , \qquad (4.1)$$

where

$$a_n>a_n^2>a_n^3>\cdots\ , \qquad (4.2)$$

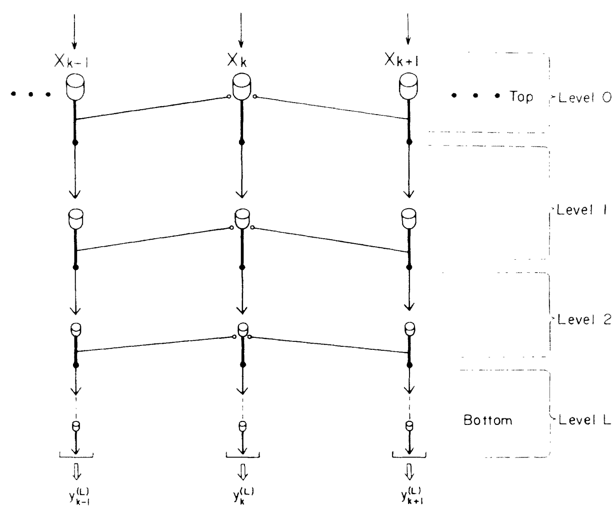

FIG. 3. Hierarchical structure of units: lateral inhibitory couplings, vertex part, and local stimuli for $0<b<1$.

from (3.5) and (3.6).

By rewriting the factor $\cos nb^l\theta$ given in the first expression of (3.4) as $\mathrm{Re}[\exp(inb^l\theta)]$, we have

$$\frac{\psi(\theta;\{a_n\},b)-1}{2}=\mathrm{Re}\sum_{n=1}^{\infty}\sum_{l=0}^{\infty}a_n^l e^{inb^l\theta}$$
$$=\mathrm{Re}\sum_{n=1}^{\infty}G_n(\theta;a_n,b)\ . \qquad (4.3)$$

Furthermore, we rewrite $G_n[=G_n(\theta;a_n,b)]$ as

$$G_n(\theta;a_n,b)=\sum_{l=0}^{\infty}e^{H_{n,l}(\theta;a_n,b)} \qquad (4.4)$$

with

$$H_{n,l}(\theta;a_n,b)=l\ln a_n+inb^l\theta\ . \qquad (4.5)$$

Recall that we have considered the energy loss arising from the phase differences between the levels in the system, and we have regarded θ as the complex variable,

$$\theta=|\theta|e^{i\delta}\ , \qquad (4.6)$$

where $|\theta|$ is an amplitude of θ and δ is a phase of θ. We replace the summation in (4.4) by an integration, and we evaluate the contributions by applying the saddle point method [13,14]. In the calculation, we use the fact that $H_{n,l}(\theta;a_n,b)$ have maxima value for $0<a_n<1$ and $0<b<1$. The maxima of $H_{n,l}(\theta;a_n,b)$ are obtained by the simultaneous equations $\partial H_{n,l}/\partial l|_{l=l_0}=0$ and $\mathrm{Re}\partial^2H_{n,l}/\partial l^2|_{l=l_0}<0$. Let l_0 be a critical value at which the simultaneous equations are satisfied. The equations read

$$\frac{1}{n\theta}=\frac{e^{i(3/2)\pi+l_0\ln b}\ln b}{\ln a_n}\ , \qquad (4.7)$$

and

$$\left.\frac{\partial^2H_{n,l}}{\partial l^2}\right|_{l=l_0}=-(\ln a_n)(\ln b)\ . \qquad (4.8)$$

From this formula, we can confirm that $\mathrm{Re}\partial^2H_{n,l}/\partial l^2|_{l=l_0}<0$, for a_n and $b<1$. Expansion of $H_{n,l}$ around $l=l_0$ and approximation of the sum by the integration lead us to

$$G_n=\tfrac{1}{2}\int_{-\infty}^{\infty}e^{H_{n,l_0}(\theta;a_n,b)-1/2(l-l_0)^2(\ln a_n)(\ln b)}dl$$
$$=\tfrac{1}{2}e^{H_{n,l_0}(\theta;a_n,b)}\left[\frac{2\pi}{(\ln a_n)(\ln b)}\right]^{1/2}\ . \qquad (4.9)$$

The exponential factor in (4.9) is calculated as

$$e^{H_{n,l_0}(\theta;a_n,b)}=\left[\frac{1}{n\theta e^{-i(\pi/2)}}\right]^{\gamma_n}e^{-\gamma_n[1-\ln\gamma_n]}\ , \qquad (4.10)$$

where $\gamma_n=\ln a_n/\ln b$, see (3.10).

Let us take a case that the factors a_n in (3.5) are given by

$$a_n = a^{(1+\varepsilon n)}, \quad (4.11)$$

where ε is a very small parameter, such that the influences are diminished as the range of couplings increases. Then the n-dependent exponent of the power law γ_n is given by

$$\gamma_n = \frac{(1+\varepsilon n)\ln a}{\ln b} = \gamma(1+\varepsilon n) . \quad (4.12)$$

With the aid of (4.11), we obtain an asymptotic form of G_n, as $\varepsilon \to 0$,

$$\lim_{\varepsilon\to 0} G_n = A_\gamma(\phi) n^{-\gamma}, \quad (4.13)$$

where a factor $A_\gamma(\phi)$ is expressed by

$$A_\gamma(\phi) = \frac{1}{2}\left[\frac{2\pi}{\gamma(\ln b)^2}\right]^{1/2} \phi^{-\gamma} e^{-\gamma(1-\ln\gamma)} \quad (\phi = \theta e^{-i(\pi/2)}) . \quad (4.14)$$

From (4.3) and (4.11)–(4.14), we finally obtain

$$\lim_{\varepsilon\to 0}\psi(\theta;\{a_n\},b) = 1 + \zeta(\gamma)\left[\frac{2\pi}{\gamma(\ln b)^2}\right]^{1/2} \times e^{-\gamma(1-\ln\gamma)}|\theta|^{-\gamma}\cos(\delta+\tfrac{3}{2}\pi)\gamma , \quad (4.15)$$

where

$$\zeta(\gamma) = \sum_{n=1}^{\infty} n^{-\gamma} , \quad (4.16)$$

and δ is the phase factor of θ [see (4.6)].

V. QUANTITATIVE BEHAVIOR OF RESPONSE FUNCTION

In Secs. I–IV we have studied asymptotic behaviors of the response functions $F_L(\theta;g_1,b)$ [see (2.13) for $n=1$] and $F_{N,L}(\theta;\{g_n\},b)$ [see (3.4) for general n]. To get the behaviors, we have utilized the smallness of the inhibitory couplings $g_n^{(l)}$ [$n=1,2,3,\ldots$, see (3.6)]. According to the analysis, the response function $F_{N,L}$ or F_L obtained approximately was identical to the Weierstrass function. This means that the approximate response functions show self-similar wave forms providing "irregular behavior."

The next problem is to make clear whether the expressions in (2.12) and (2.13) give similar irregular behaviors or not. The problem sheds light on an origin of the irregular behavior of the response functions. To this end, we first study the response function $F_{N,L}$ quantitatively without any approximation, and we compare it with the corresponding analytic form obtained approximately. Specifically, we investigate wave forms obtained from $F_{N,L}$ for $b>1$ [see (3.3)] because the case for $b>1$ cannot be studied by the saddle point method. For simplicity, we concentrate our calculation on a function $\tilde{F}_{1,L}(\theta)$ defined below, corresponding to $F_{1,L}(\theta;g_1^{(1)},b)$,

$$\tilde{F}_{1,L}(\theta) = \prod_{l=0}^{L_{\max}} [1+2g_1^{(l)}\cos b^l\theta] . \quad (5.1)$$

By introducing a parameter C by $Ca_l = g_1^{(l)}$, $\tilde{F}_{1,L}(\theta)$ becomes as follows:

$$\tilde{F}_{1,L}(\theta) = \prod_{l=0}^{L_{\max}} [1+2Ca^l\cos b^l\theta] \quad (5.2)$$

$$\simeq 1 + 2C\sum_{l=0}^{L_{\max}} a^l\cos b^l\theta , \quad (5.3)$$

where $L_{\max}$ stands for an upper limit of L. The question whether the couplings are weak or strong depends on the magnitude of C. We call a case with $C=0.1$ a weak coupling one, whereas we regard a case with $C=0.8$ as a strong coupling one.

For both cases, we obtain quantitative results for

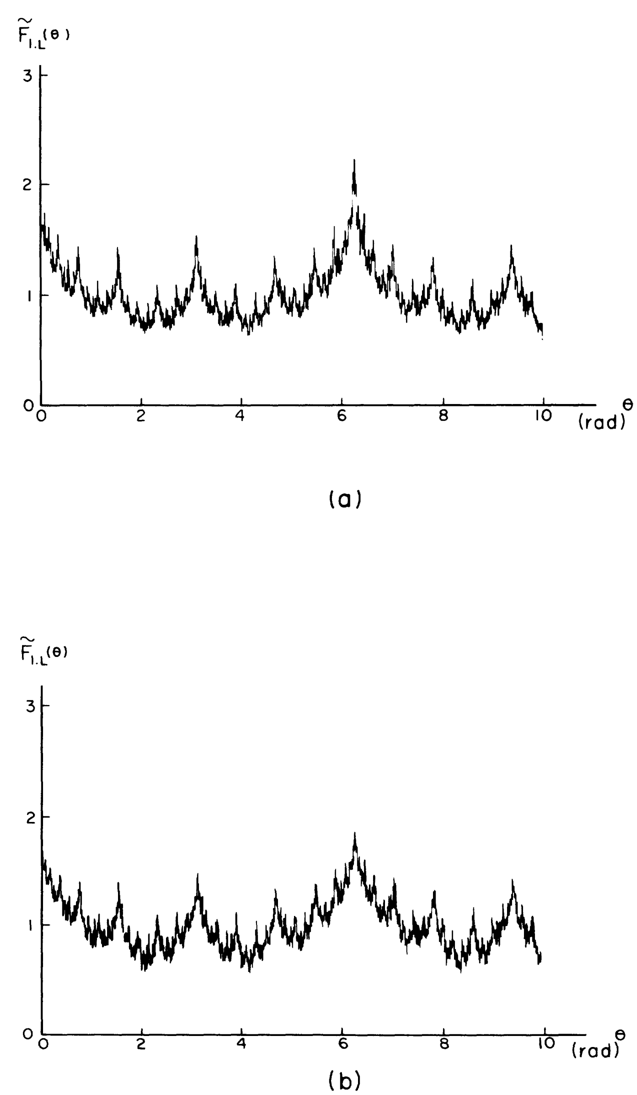

FIG. 4. (a) Behavior of $\tilde{F}_{1,L}(\theta)$ given by (5.1) or (5.2). (b) Behavior of $\tilde{F}_{1,L}(\theta)$ given by (5.3). The quantitative behaviors are for a weak coupling case ($C=0.1$, $a=0.8$) and $L_{\max}=50$, $b=2$. Here θ is expressed in radians and is taken to be a real variable.

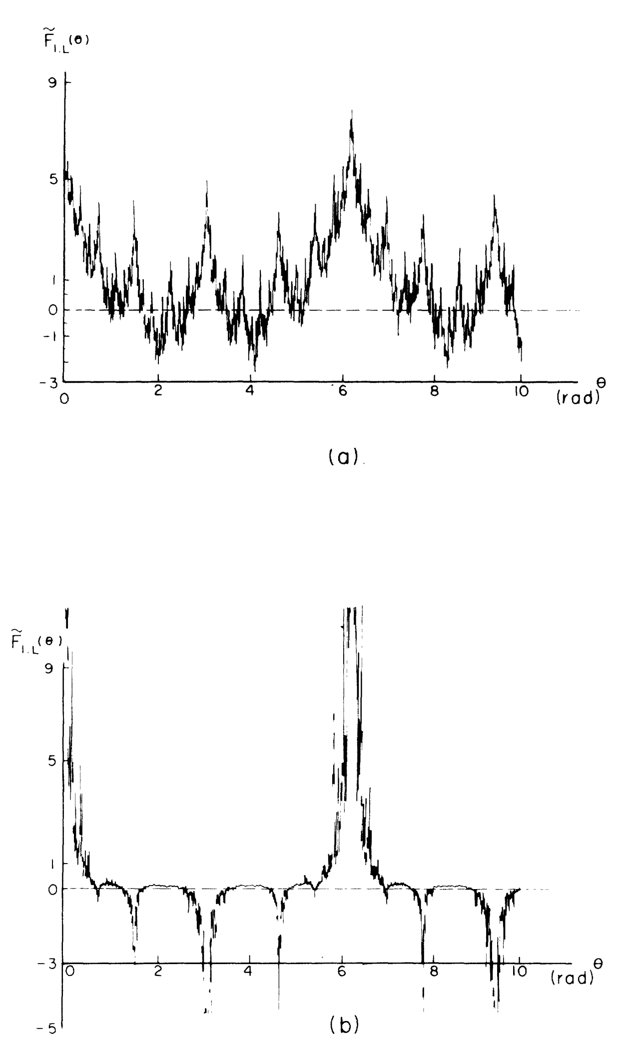

FIG. 5. (a) Behavior of $\tilde{F}_{1,L}(\theta)$ given by (5.1) or (5.2). (b) Behavior of $\tilde{F}_{1,L}(\theta)$ given by (5.3). The quantitative behaviors are for a strong coupling case ($C=0.8$, $a=0.8$) and $L_{\max}=100$, $b=2$. Here θ is expressed in radians and is taken to be a real variable.

$\tilde{F}_{1,L}(\theta)$ from the exact expression given in (5.1) and the corresponding approximate expression (5.3). The results are given for two bifurcated fibers ($b=2$).

Figures 4(a) and 4(b) show the behaviors of (5.1) or (5.2) and (5.3) for the weak coupling case ($C=0.1$, $a=0.8$), respectively. The behaviors are in good agreement with the results obtained. For the strong coupling case ($C=0.8$, $a=0.8$), however, there is discrepancy between the results for (5.1) or (5.2) and (5.3), as is easily seen in Figs. 5(a) and 5(b), respectively.

Based on numerical calculations, it is found that the irregular behavior is obtained for the weak coupling case from the exact response function (5.1) as well as (5.2). On the other hand, the irregular behaviors are localized, and "dominant modes" relevant to the strong coupling case are obtained. Expressed differently, the irregular behaviors are confirmed to be intrinsic for the weak coupling case.

VI. CONCLUDING REMARKS

In this paper, we proposed a model of information processing of a complex system. Specifically, we studied properties of the response function representing a visual pattern processing. The model system was composed of a set of units forming multiple layers. The layers were constructed by generalizing the usual lateral neural inhibitions between the units. The connections between the units were assumed to be interconnected vertically by successive bifurcations and influenced horizontally by lateral inhibitory couplings. It was found that a local response function was expressed approximately by the Weierstrass function and the behavior was specified asymptotically by a power law The exponent of the power law represents the fractal dimension determined by the interconnections and the lateral inhibitory couplings.

The behavior was analytically obtained by applying the saddle point method [13,14] for Eq. (4.4). In the evaluation, we have used the fact that $H_{n,l}(\theta;a_n,b)$ have maxima for $0<a_n<1$ and $0<b<1$, as $l\to\infty$. This case ($b<1$) represents special unbifurcated connections, and the parameter b plays the role of a weight factor to the local stimuli (see Fig. 3). For the case $b>1$ we could not apply the saddle point method. For this reason, with the aid of numerical analysis (Sec. V), we studied behaviors of the exact and the approximate expressions for $\tilde{F}_{1,L}(\theta)$, and confirmed that behaviors similar to the Weierstrass function were obtained for a weak coupling case. On the other hand, the response function for a strong coupling case showed no self-similar wave form. This means that the behaviors are not irregular. The irregular behaviors of the weak coupling case are localized to dominant modes relevant to the strong coupling case. Expressed differently, in the weak coupling case the hierarchical connections lead us to the response function characterized by the fractal dimension, as it should be. But in the strong coupling case the hierarchical connections are not relevant to the self-similar behavior of the response function.

Generally, the main functions or response behaviors of neural networks are understood by geometrical structures of the connections among the units. Furthermore, some of the features can be simulated by the analog electrical circuits [8,12]. From an architectural point of view, the present model supports the idea that information processing is relevant for settling or modifying the connections of the neural networks. The idea may be extended to the other functions of biological systems in which the self-similar structures are essential features as seen in arteries, veins, the bronchia of the lung, and so on. In the geophysical system [15], if we regard the data observed by the seismometer as successive stimuli-response propagations, the present results of the response function suggest a hint to the phenomena.

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